The Structure of Commutative Algebra Has the Only One Regular Maximal Ideal

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ABSTRACT

The purpose of this paper is to characterize the commutative algebra having the only one regular maximal ideal. We will show that the following two statements are equivalent:

(1) There exists a nonzero idempotent $v \in A$ and a F-algebra homomorphism $f: A \rightarrow F$ such that $xy=f(x)f(y)v$, $\forall x, y \in A$.

(2) *1.* There exists a nonzero idempotent $u \in A$.

2. There exists a regular maximal ideal M such that $MA=0$.

3. $A/M \cong F$.

We first review some definitions and properties, which are used in the proof of conclusion and also we give an example to illustrate this result.

I want to dedicate this paper to my wife Hsieh Mei-Ling, I love her with all my heart.

I. Preliminaries

In this section, R denotes a commutative ring with identity.

Definition 1: A commutative R-algebra A is a unital R-module, that is also a commutative ring such that

$$r(ab) = (ra)b = a(rb) \qquad \forall a,b \in A, r \in R.$$

If there is an element $e \in A$ such that $ae=a=ea \quad \forall a \in A$, then A is called R-algebra with identity e.

Definition 2: A R-submodule I of A is called an ideal of A. If I is stable under multiplication and $AI \subset I$. Here AI denotes the set of all elements ab, $\forall a \in A, b \in I$.

Definition 3: An element $e \neq 0$ in A is called an idempotent, if $e^2 = e$.

Definition 4: A proper ideal I of a commutative R-algebra A without identity is called a regular ideal, if there exists an element $a \in A$ such that $ar-r \in I$, $\forall r \in A$. Call a regular maximal ideal, if it is a maximal ideal or equivalently the quotient ring A/I is a field.

Definition 5: The intersection of all regular maximal ideals of a R-algebra A is the radical of A, denote Rad A.

Definition 6: Let A and B be two R-algebras, A R-module homomorphism f: A→B is called a R-algebra homomorphism, if f(ab)=f(a)f(b), $\forall$ a, $b \in A$.

A bijetive homomorphism is called an isomorphism. A surjetive homomorphism is called an epimorphism. Note that e and e' denote the identities of A and B, we will always assume that f(e)=e'.

Through this section, A denotes a commutative R-algebra without identity.

Lemma 1: If I is an ideal of A containing a regular ideal K, then I is a regular ideal.

proof: Since K is a regular ideal, there exists $a \in A$ such that $ax - x \in K \subset I, \forall x \in A$. Thus I is a regular ideal.

Lemma 2: Every regular ideal of A is contained in a regular maximal ideal.

proof : Let I be a proper regular ideal of A. Then there exists $a \in A$ such that $ar - r \in I, \forall r \in A$. Set Σ={K | K is a regular ideal of A such that $a \notin K$ and I$\subseteq$K} Then $\Sigma \neq \Phi$, since $I \in \Sigma$. (If $a \in I$ then $ar \in I$ and I=A, hence $a \notin I$). Σ is a poset with set inclusion. Let $\{I_\alpha\}$ be any chain in Σ, Then $\underset{\alpha}{U} I_\alpha$ is an upper bound of the chain. We claim that $\underset{\alpha}{U} I_\alpha \in \Sigma$. It is clear that $\underset{\alpha}{U} I_\alpha$ is an ideal.

Sine$\{I_\alpha\}$ is a chain, $I \subset \underset{\alpha}{U} I_\alpha$ and $a \notin \underset{\alpha}{U} I_\alpha$, hence $\underset{\alpha}{U} I_\alpha \in \Sigma$. By Zorn's lemma,

Σ has a maximal element M, then M is an ideal of A, $a \notin$M and I$\subset$M. We claim that M is a maximal ideal. Let D be a regular ideal of A and M$\subsetneq$D$\subset$A, Then $a \in$D. Since $ar - r \in M \cap D \quad \forall r \in A$, hence r$\in$D, thus D=A. M is a regular maximal ideal of A.

Lemma 3: If f: A$\rightarrow$B is a homomorphism, then Kerf and A/kerf are R-algebras.

proof: Trivial.

Lemma 4: If f:A$\rightarrow$B is an epimorphism, then $A/Kerf \cong B$.

proof: Trivial.

Lemma 5: Let A be a commutative R-algebra and u nonzero idempotent. Then u does not belong to the radical of A.

proof : We first show that $u \neq au - a \quad \forall a \in A$. If $u=au-a$ then $a+u=au$ and $ua + u^2 = u(a+u) = u \cdot au = au \Rightarrow u^2 = u = 0$, a contradiction.

Let $S = \{ux - x | \forall x \in A\}$, We claim that S is a regular ideal of A.

For any $x, y \in A$, $r \in R$ $b \in A$

$$(ux - x) - (uy - y) = u(x - y) - (x - y) \in S$$

$$r(ux - x) = u(rx) - rx \in S$$

and

$$b(ux - x) = u(bx) - bx \in S$$

Thus S is a regular ideal. By lemma 2, there exists a regular maximal ideal M such that $S \subset M$. We claim that $u \notin M$. If $u \in M$ and M is a regular maximal ideal, then $\forall x \in A, ux - x \in S \subset M, ux \in M$ implies $x \in M$. Thus $M=A$, a contradition.

II : Main Result

Now, we give a characterization of a commutative F-algebra without identity having exactly one regular maximal ideal. F denote a field.

Main theorem:

Let A be a commutative F-algebra. Then the following two conditions are equivalent :

1. There exists $v \in A$ such that $v \neq 0 \neq v^2$ and for any $x.y$ in A, $x \cdot y = f(x)f(y)v$, where f: $A \to F$ is a F-algebra homomorphism such that $f(v)=1$.

2. (1) There exists a non-zero idempotent $u \in A$.

   (2) There exists exactly one regular maximal ideal $M \subset A$.
   Furthermore, $\forall m \in M, x \in A, mx = 0$

   (3) $A/M \cong F$.

proof: ($1 \Rightarrow 2$) Since $v \neq 0$ and any x, y in A, xy=f(x)f(y)v, f(v)=1, then $v \cdot v = f(v)f(v)v = v$, hence v is nonzero idempotent in A.

(2) Let $M=\{m\in A \mid mx=0,\forall x\in A\}$. We claim that M is a regular ideal. For any m, n∈M, a, x∈A, r∈F, we have

$$(m-n)x=mx-nx=0, m-n\in M$$

$$(mn)x=m(nx)=m\cdot 0=0, mn\in M$$

$$(am)x=a(mx)=a\cdot 0=0, am\in M$$

$$(rm)x=r(mx)=r\cdot 0=0, rm\in M$$

For any x, y∈A, $(vx-x)\cdot y=(f(v)f(y)v-x)y=f(v)f(x)f(v)\,f(y)v-f(x)f(y)v=0$

hence $vx-x\in M$, Thus M is a regular ideal of A.

We claim that M is a regular maximal ideal. If I is any regular ideal, then there exists a∈A such that $ax-x\in I$, $\forall x\in A$. Since I is a proper ideal of A, there exists b∈A and b∉I such that $a\cdot b-b=f(a)f(b)v-b\in I$, hence $f(a)f(b)\ v\notin I$ implies v∉I. If there exists a∈I such that av≠0, then $f(a)v=f(a)f(v)v=av\in I$; Since F is a field, hence v∈I, a contradiction. Thus xv=0, $\forall x\in I$ implies I⊆ M. By lemma 1, we know that any ideal containing M must be regular ideal, hence I=M. Therefore M is the only regular maximal ideal.

(3) Consider that f: A→F is an epimorphism. Since f(v)=1and $f(rv)=rf(v)=r,\forall r\in F$; we claim that Ker f=M. For any m∈ M,

mv=f(m)f(v)v=f(m)v=0 implies f(m)=0

that is $m\in Ker f$ and $M\subset ker f$. But M is the only regular maximal ideal, hence $Ker f=M$. By lemma 4,we have A/M≅F.

Conversely, let u be a nonzero idempotent in A.Put I=$\{ux-x|\forall x\in A\}$, I is a regular ideal and $I \subset M$, since M is the only one regular maximal ideal. By lemma 5, we know that $Rad\,A=M$, $u \notin RadA = M$.

Let i: A/M→F be an isomorphism and e: A→A/M the natural epimorphism. Put f=ioe, it is clear that f is an epimorphism and f(u)=1, since u+M is the identity of A/M.

$$\begin{tikzcd} A \arrow[r, "e"] & A/M \arrow[dl, "i"] \\ F \arrow[u, "f"] & \end{tikzcd}$$

We claim that A=M$\oplus$ Fu. For any a$\in$A, there exists

$$r=i(a+M)=f(a) \text{ such that } f(a-ru)=f(a)-rf(u)=r-r=0,\ a-ru\in M.$$

We claim that M$\cap$Fu=$\{0\}$. If x$\in$M$\cap$Fu then x$\in$M and x$\in$Fu, hence

$$x=ru,\ r\in F. \text{ Since } u\notin M,\ r=0 \text{ and } x=0.$$

We show that $x \cdot y = f(x)f(y)u, \forall x, y \in A$. For any $x,y\in A$, we have $x = m_1 + r_1u, y = m_2 + r_2u$, where $r_1 = f(x), r_2 = f(y)$. Since $mx = o, \forall x \in A, m \in M$, we have

$$x \cdot y = (m_1 + r_1u)(m_2 + r_2u) = m_1m_2 + r_1um_2 + r_2um_1 + r_1r_2u^2 = r_1r_2u = f(x)f(y)u.$$

The following example is to illustrate the existence of such algebra.

Example: let V be an inner product vector space over real field R and u a unit vector. Define the multiplication "o" by $mon = (m \cdot u)(n \cdot u)u, \forall m, n \in V$.

It is easily check the (V,+,o) is a commutative ring, We claim that (V,+,o) has an identity.

If dim V=1 then V is generated by u; that is, for any m$\in$V there exists $r \in R$ such that m=ru; Then

$$mou = (m \cdot u)(u \cdot u)u = (ru \cdot u)(u \cdot u)u = ru = m \quad \text{since} \quad u \cdot u = 1,$$

hence u is the identity in V.

If dim $V = k \geq 2$, then V has no identity. Assume that e is an identity of V, then

$$e = eoe = (e \cdot u)(e \cdot u)u \in [u]$$

where [u] denote the ideal generated by u. Hence $e=ru$, for some r$\in R$.Since dim $V = k \geq 2$, there exists $m \notin [u]$, then

$$m = moe = (m \cdot u)(e \cdot u)u = (m \cdot u)ru \in [u], \text{ a contradiction.}$$

Therefore V has no identity.

Define $M = \{m \in V | m \cdot u = 0\}$. We claim that M is a regular maximal ideal of V. Clearly, it is an ideal.

$$(m\text{-}mou) \cdot u = m \cdot u - (m \cdot u)(u \cdot u)(u \cdot u) = 0 \quad \forall m \in V$$

hence m-mou$\in$M, therefore M is a regular ideal.

Define f: $V/M \to R$ by $f(v+M) = v \cdot u, \forall v \in V$. We show that f is well-defined. If $a+M=b+M$ then $a\text{-}b \in M$, $(a-b) \cdot u = (a \cdot u) - (b \cdot u) = o$ implies $a \cdot u = b \cdot u \ \ \forall a,b \in V$.

Now we claim that f is isomorphism. Clearly f is a homomorphism;

$$f(ru + M) = ru \cdot u = r \quad \forall r \in R, ru + M \in V/M$$

and

$$\ker f = \{a + M | a \cdot u = 0\} = \{a + M | a \in M\} = \{M\}.$$

Thus

$$V / M \cong R$$

Since R is a field, hence M is the only one maximal ideal.

Reference

(1) Lambek: Lectures on Rings and Modules; McCrill University (1966) (27～31, 72～79).

(2) Carl Faith: Algebra I Rings: Modules and Categories; Spring Uerlag (1984), (419～440).

(3) Dennis B. Ames: An introduction to abstract Algebra; Fullerton, California, (1969). (162～177, 341～350).

(4) M.F. Atiyah, frs, I.G. Macdonald: Introduction to Commutative Algebra, Oxford, (1969), (1～10, 24～39).

(5) Jacob K.Goldhaber and Gertrude Ehrlich: Algebra; University of Maryland.

(6) D.G. Northcott: Ideal Theory; Cambridge, (1953).

(7) O. Zariski and P. Samuel: Commutative Algebra I, II; Van Nostrand; Princeton, (1960).

(8) Burton: A first course in rings and ideals; Addision-Wesley. (1970)

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在交換代數中僅有一個極大理想結構性的探討

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摘　要

本論文是探討在可交換 F-algebra 下，僅有一個正則極大理想必須具備的結構。

全文中，首先建立一些預備定理；然後提出兩等價條件，加以證明說明出只有一個正則極大理想的形態，最後再提出一個實例，說明其存在性。

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